

WHITTAKER RATIONAL STRUCTURES AND SPECIAL VALUES OF THE ASAI L -FUNCTION

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Dedicated to Jim Cogdell on the occasion of his 60th birthday

ABSTRACT. Let F be a totally real number field and E/F a totally imaginary quadratic extension of F . Let Π be a cohomological, conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Under a certain non-vanishing condition we relate the residue and the value of the Asai L -functions at $s = 1$ with rational structures obtained from the cohomologies in top and bottom degrees via the Whittaker coefficient map. This generalizes a result in Eric Urban's thesis when $n = 2$, as well as a result of the first two named authors, both in the case $F = \mathbb{Q}$.

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1. INTRODUCTION

Let F be a totally real number field and E/F a totally imaginary quadratic extension of F with non-trivial Galois involution τ . Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. One can associate two Asai L -functions over F , denoted $L(s, \Pi, \mathrm{As}^+)$ and

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$L(s, \Pi, \text{As}^-)$. These are Langlands L -functions attached to representations of the L -group of GL_n/E , and the Rankin-Selberg product of Π with Π^τ factors as

$$(1.1) \quad L(s, \Pi \times \Pi^\tau) = L(s, \Pi, \text{As}^+) \cdot L(s, \Pi, \text{As}^-).$$

In this paper we consider representations Π that arise by stable quadratic base change from an automorphic representation π of the unitary group H over F . In particular, Π is *conjugate self-dual*:

$$\Pi^\vee \cong \Pi^\tau.$$

We have an equality of partial L -functions

$$L^S(s, \Pi, \text{As}^{(-1)^n}) = L^S(s, \pi, \text{Ad}),$$

where Ad is the adjoint representation of the L -group of H . (The other Asai L -function equals the L -function of π with respect to the twist of Ad by the character corresponding to E/F .)

Since $\Pi^\tau \cong \Pi^\vee$, the Rankin-Selberg L -function on the left-hand side of (1.1) has a simple pole at $s = 1$ and the assumption that Π is a base change from a unitary group implies that this pole arises as the pole of $L(s, \Pi, \text{As}^{(-1)^{n-1}})$ at $s = 1$. Moreover, $L(s, \Pi, \text{As}^{(-1)^n})$ is holomorphic and non-vanishing at $s = 1$. This applies in particular if Π is cohomological and conjugate-dual: Then it is known that Π is automatically a base change from some unitary group H , and moreover $s = 1$ is a critical value of $L(s, \Pi, \text{As}^{(-1)^n})$.

Hypothetically, $L(s, \Pi, \text{As}^{(-1)^{n-1}}) = \zeta(s) L(s, M^b(\Pi))$ for some motive $M^b(\Pi)$ (which we do not specify here), where $\zeta(s)$ is the Riemann zeta function. One of the main goals of this note is to relate the residue at $s = 1$ of $L(s, \Pi, \text{As}^{(-1)^{n-1}})$, which under the above hypothesis can be interpreted as a *non-critical* special value of the L -function of $M^b(\Pi)$, to a certain cohomology class attached to Π , of the adelic “locally symmetric” space $\mathcal{S}_E = \text{GL}_n(E) \backslash \text{GL}_n(\mathbb{A}_E) / A_G K_\infty$.

In fact, Π contributes to the cohomology of \mathcal{S}_E (with suitable coefficients) in several degrees. For each degree q , where it contributes, one can define a rational structure on the q -th $(\mathfrak{m}_G, K_\infty)$ -cohomology of Π_∞ , which measures the difference between the global cohomological rational structure and the one defined using the Whittaker-Fourier coefficient. We call it the *Whittaker comparison rational structure* (CRS) of degree q and denote it by \mathbb{S}_q . Let b and t be the minimal and maximal degrees, respectively, where the $(\mathfrak{m}_G, K_\infty)$ -cohomology of Π_∞ is non-zero. Roughly speaking, our main result is that under a suitable local non-vanishing assumption, $\text{Res}_{s=1} L(s, \Pi, \text{As}^{(-1)^{n-1}})$ (resp., $L(1, \Pi, \text{As}^{(-1)^n})$) spans, under suitable normalization, the one-dimensional spaces \mathbb{S}_t (resp., \mathbb{S}_b) over the field of definition of Π . The precise results are stated in Theorems 6.4 and 7.1 in the body of the paper. In the case $n = 2$ and $F = \mathbb{Q}$ such results had been proved in the theses of Eric Urban and Eknath Ghate, respectively [Urb95, Gha99]. The results here sharpen some of the main results of [GH13] (for $F = \mathbb{Q}$). We are hopeful that the pertinent non-vanishing assumption will be settled in the near future using the method recently developed by Binyong Sun.

The result for the top degree cohomology turns out to be a rather direct consequence of the well-known relation between the residue of the Asai L -function at $s = 1$ and the period integral over $\text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}) / A_G$ [Fli88]. This is a twisted analogue of the realization of the Petersson inner product in the Whittaker model, due to Jacquet–Shalika [JS81].

The two results for the top and bottom degrees are linked by Poincaré duality and the relation (1.1). More precisely, one can relate $\text{Res}_{s=1} L(s, \Pi \times \Pi^\vee)$ to a suitable pairing between \mathbb{S}_q and \mathbb{S}_{d-q} in any degree q (where d is the dimension of \mathcal{S}_E). Once again, this is a simple consequence of the aforementioned result of Jacquet–Shalika. Although we prove this Poincaré duality result only for (cohomological) conjugate self-dual representations of $\text{GL}_n(\mathbb{A}_E)$, the methodology is applicable for any cohomological representation over any number field. In fact, a result of a similar flavor (in this generality) was obtained independently about the same time by Balasubramanyam–Raghuram [RB14].

We also note that A. Venkatesh has recently proposed a conjecture relating the rational structure of the contribution of Π to cohomology (in all degrees) to the K -theoretic regulator map of a hypothetical motive attached to Π . We hope that the precise statement of the conjecture will be available soon. At any rate, our results seem to be compatible with this conjecture.

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2. NOTATION AND CONVENTIONS

2.1. Number fields and the groups under consideration. Throughout the paper, let F be a totally real number field and E/F denotes a totally imaginary quadratic extension of F with non-trivial Galois involution τ . The discriminant of F (resp. E) is denoted D_F (resp. D_E). We let I_E be the set of field embeddings $E \hookrightarrow \mathbb{C}$ and the ring of integers (resp., adeles) of E by \mathcal{O}_E and \mathbb{A}_E . (Similar notation is used for the field F .) We fix a non-trivial, continuous, additive character $\psi : E \backslash \mathbb{A}_E \rightarrow \mathbb{C}^\times$.

We fix an integer $n \geq 1$ and use G to denote the general linear group GL_n viewed as a group scheme over \mathbb{Z} .

We denote by T the diagonal torus in G , by \mathcal{P} the subgroup of G consisting of matrices whose last row is $(0, \dots, 0, 1)$, and by U the unipotent subgroup of upper triangular matrices in G .

For brevity we write $G_\infty = R_{E/\mathbb{Q}}(G)(\mathbb{R})$, where $R_{E/\mathbb{Q}}$ stands for the restriction of scalars from E to \mathbb{Q} . We write A_G for the group of positive reals \mathbb{R}_+ embedded diagonally in the center of G_∞ . Thus, $G(\mathbb{A}_E) \cong G(\mathbb{A}_E)^1 \times A_G$ where $G(\mathbb{A}_E)^1 := \{g \in G(\mathbb{A}_E) : |\det g|_{\mathbb{A}_E^*} = 1\}$.

Let K_∞ be the standard maximal compact subgroup of G_∞ isomorphic to $U(n)^{[F:\mathbb{Q}]}$. We set $\mathfrak{g}_\infty = \text{Lie}(G_\infty)$, $\mathfrak{k}_\infty = \text{Lie}(K_\infty)$, $\mathfrak{a}_G = \text{Lie}(A_G)$, $\mathfrak{p} = \mathfrak{g}_\infty / \mathfrak{k}_\infty$, $\mathfrak{m}_G = \mathfrak{g}_\infty / \mathfrak{a}_G$ and $\tilde{\mathfrak{p}} = \mathfrak{p} / \mathfrak{a}_G$. Let $d := \dim_{\mathbb{R}} \mathfrak{m}_G - \dim_{\mathbb{R}} \mathfrak{k}_\infty$. The choice of measures is all-important in all results of this kind. Our choices are specified in Sections 5.2 and 6.3.

2.2. Coefficient systems. We fix an irreducible, finite-dimensional, complex, continuous algebraic representation E_μ of G_∞ . It is determined by its highest weight $\mu = (\mu_\iota)_{\iota \in I_E}$ where for each ι , $\mu_\iota = (\mu_{1,\iota}, \dots, \mu_{n,\iota}) \in \mathbb{Z}^n$ with $\mu_{1,\iota} \geq \mu_{2,\iota} \geq \dots \geq \mu_{n,\iota}$. We assume that E_μ is conjugate self-dual, i.e., $E_\mu^\tau \cong E_\mu^\vee$, or, in other words, that

$$\mu_{j,\tau(\iota)} + \mu_{n+1-j,\iota} = 0, \quad \iota \in I_E, \quad 1 \leq j \leq n.$$

2.3. Cuspidal automorphic representations. Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_E)$. We shall assume that Π is conjugate self-dual, i.e., $\Pi^\tau \cong \Pi^\vee$. We say that Π is *cohomological* with respect to E_μ , if $H^*(\mathfrak{g}_\infty, K_\infty, \Pi \otimes E_\mu) \neq 0$. We refer to Borel–Wallach [BW00], I.5, for details concerning $(\mathfrak{g}_\infty, K_\infty)$ -cohomology.

Throughout the paper we assume that Π is a conjugate self-dual, cuspidal automorphic representation which is cohomological with respect to E_μ . (However, this hypothesis is not used before the proof of Theorem 5.3.) Denote the Petersson inner product on $\Pi \times \Pi^\vee$ by

$$(\varphi, \varphi^\vee)_{\mathrm{Pet}} := \int_{G(E) \backslash G(\mathbb{A}_E)^1} \varphi(g) \varphi^\vee(g) dg.$$

We write $\mathcal{W}^\psi(\Pi)$ for the Whittaker model of Π with respect to the character

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

Similarly for $\mathcal{W}^{\psi^{-1}}(\Pi^\vee)$. Let

$$W^\psi : \Pi \rightarrow \mathcal{W}^\psi(\Pi)$$

be the realization of Π in the Whittaker model via the ψ -Fourier coefficient, namely

$$W^\psi(\varphi) = (\mathrm{vol}(U(E) \backslash U(\mathbb{A}_E)))^{-1} \int_{U(E) \backslash U(\mathbb{A}_E)} \varphi(ug) \psi(u)^{-1} du.$$

Analogous notation is used locally.

2.4. Pairings of $(\mathfrak{m}_G, K_\infty)$ -cohomology spaces. Suppose that ρ and ρ^* are two irreducible $(\mathfrak{g}_\infty, K_\infty)$ -modules which are in duality and let (\cdot, \cdot) be a non-degenerate invariant pairing on $\rho \times \rho^*$. For $p + q = d$, let us define a pairing

$$\mathbb{K}_{(\rho, \rho^*, (\cdot, \cdot))}^{\mathrm{coh}, p} : H^p(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \rho^* \otimes E_\mu^\vee) \rightarrow (\wedge^d \tilde{\mathfrak{p}})^*$$

as follows: Recall that

$$\begin{aligned} H^p(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu) &\xrightarrow{\sim} \mathrm{Hom}_{K_\infty}(\wedge^p \tilde{\mathfrak{p}}, \rho \otimes E_\mu), \\ H^q(\mathfrak{m}_G, K_\infty, \rho^* \otimes E_\mu^\vee) &\xrightarrow{\sim} \mathrm{Hom}_{K_\infty}(\wedge^q \tilde{\mathfrak{p}}, \rho^* \otimes E_\mu^\vee). \end{aligned}$$

Suppose that $\tilde{\omega} \in \mathrm{Hom}_{K_\infty}(\wedge^p \tilde{\mathfrak{p}}, \rho \otimes E_\mu)$ and $\tilde{\eta} \in \mathrm{Hom}_{K_\infty}(\wedge^q \tilde{\mathfrak{p}}, \rho^* \otimes E_\mu^\vee)$ represent $\omega \in H^p(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$ and $\eta \in H^q(\mathfrak{m}_G, K_\infty, \rho^* \otimes E_\mu^\vee)$ respectively. The cap product

$$\tilde{\omega} \wedge \tilde{\eta} \in \mathrm{Hom}_{K_\infty}(\wedge^d \tilde{\mathfrak{p}}, \rho \otimes \rho^* \otimes E_\mu \otimes E_\mu^\vee)$$

together with the pairing on $\rho \times \rho^*$ and the canonical pairing on $E_\mu \otimes E_\mu^\vee$, define an element

$$\mathbb{K}_{(\rho, \rho^*, (\cdot, \cdot))}^{\mathrm{coh}, p}(\omega, \eta) \in (\wedge^d \tilde{\mathfrak{p}})^*.$$

Note that $(\wedge^d \tilde{\mathfrak{p}})^*$ is canonically isomorphic to the space of invariant measures on $G_\infty/A_G K_\infty$.

2.5. **Locally symmetric spaces over E .** Recall the adelic quotient

$$\mathcal{S}_E := G(E) \backslash G(\mathbb{A}_E)^1 / K_\infty.$$

We can view \mathcal{S}_E as the projective limit $\mathcal{S}_E = \varprojlim_{K_f} \mathcal{S}_{E, K_f}$ where

$$\mathcal{S}_{E, K_f} = G(E) \backslash G(\mathbb{A}_E)^1 / K_\infty K_f$$

and K_f varies over the directed set of compact open subgroups of $G(\mathbb{A}_{E, f})$ ordered by opposite inclusion. Note that each \mathcal{S}_{E, K_f} is a orbifold of dimension $d = n^2[F : \mathbb{Q}] - 1$.

A representation E_μ as in §2.2 defines a locally constant sheaf \mathcal{E}_μ on \mathcal{S}_E whose espace étalé is $G(\mathbb{A}_E)^1 / K_\infty \times_{G(E)} E_\mu$ with the discrete topology on E_μ . We denote by $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$ and $H_c^q(\mathcal{S}_E, \mathcal{E}_\mu)$ the corresponding spaces of sheaf cohomology and sheaf cohomology with compact support, respectively. They are $G(\mathbb{A}_{E, f})$ -modules. We have

$$H^q(\mathcal{S}_E, \mathcal{E}_\mu) \cong \varinjlim_{K_f} H^q(\mathcal{S}_{E, K_f}, \mathcal{E}_\mu)$$

and

$$H_c^q(\mathcal{S}_E, \mathcal{E}_\mu) \cong \varinjlim_{K_f} H_c^q(\mathcal{S}_{E, K_f}, \mathcal{E}_\mu),$$

where the maps in the inductive systems are the pull-backs (Rohlf's [Roh96] Cor. 2.12 and Cor. 2.13). For our purposes we will only use this result to save notation (or to avoid an abuse of notation): we could have simply worked throughout with the inductive limits of cohomologies. (In fact, in [Clo90] $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$ is simply defined as $\varinjlim_{K_f} H^q(\mathcal{S}_{E, K_f}, \mathcal{E}_\mu)$.)

Let $H_{\text{cusp}}^q(\mathcal{S}_E, \mathcal{E}_\mu)$ be the $G(\mathbb{A}_{E, f})$ -module of cuspidal cohomology, being defined as the $(\mathfrak{m}_G, K_\infty)$ -cohomology of the space of cuspidal automorphic forms. As cusp forms are rapidly decreasing, we obtain an injection

$$\Delta^q : H_{\text{cusp}}^q(\mathcal{S}_E, \mathcal{E}_\mu) \hookrightarrow H_c^q(\mathcal{S}_E, \mathcal{E}_\mu).$$

3. INSTANCES OF ALGEBRAICITY

3.1. **An action of $\text{Aut}(\mathbb{C})$.** Let ν be a smooth representation of either $G(\mathbb{A}_{E, f})$ or $G(E_w)$ for a non-archimedean place w of E , on a complex vector space W . For $\sigma \in \text{Aut}(\mathbb{C})$, we define the σ -twist $\sigma\nu$ following Waldspurger [Wal85], I.1: If W' is a \mathbb{C} -vector space which admits a σ -linear isomorphism $\phi : W \rightarrow W'$ then we set

$$\sigma\nu := \phi \circ \nu \circ \phi^{-1}.$$

This definition is independent of ϕ and W' up to equivalence of representations. One may hence always take $W' := W \otimes_\sigma \mathbb{C}$.

On the other hand, let $\nu = E_\mu$ be a highest weight representation of G_∞ as in §2.2. The group $\text{Aut}(\mathbb{C})$ acts on I_E by composition. Hence, we may define ${}^\sigma E_\mu$ to be the irreducible representation of G_∞ , whose local factor at the embedding ι is $E_{\mu_{\sigma^{-1}\iota}}$, i.e., has highest weight $\mu_{\sigma^{-1}\iota}$. As a representation of the diagonally embedded group $G(E) \hookrightarrow G_\infty$, ${}^\sigma E_\mu$ is isomorphic to $E_\mu \otimes_\sigma \mathbb{C}$, cf. Clozel [Clo90], p. 128. Moreover, we obtain

Proposition 3.1. *For all $\sigma \in \text{Aut}(\mathbb{C})$, ${}^\sigma\Pi_f$ is the finite part of a cuspidal automorphic representation ${}^\sigma\Pi$ which is cohomological with respect to ${}^\sigma E_\mu$. The representation ${}^\sigma\Pi$ is conjugate self-dual.*

Proof. See [Clo90], Thm. 3.13. (Note that an irreducible $(\mathfrak{g}_\infty, K_\infty)$ -module is cohomological if and only if it is regular algebraic in the sense of [loc. cit.].) The last statement is obvious. \square

3.2. Rationality fields and rational structures. Recall also the definition of the rationality field of a representation (e.g., [Wal85], I.1). If ν is any of the representations considered above, let $\mathfrak{S}(\nu)$ be the group of all automorphisms $\sigma \in \text{Aut}(\mathbb{C})$ such that ${}^\sigma\nu \cong \nu$:

$$\mathfrak{S}(\nu) := \{\sigma \in \text{Aut}(\mathbb{C}) \mid {}^\sigma\nu \cong \nu\}.$$

Then the *rationality field* $\mathbb{Q}(\nu)$ is defined as the fixed field of $\mathfrak{S}(\nu)$,

$$\mathbb{Q}(\nu) := \{z \in \mathbb{C} \mid \sigma(z) = z \text{ for all } \sigma \in \mathfrak{S}(\nu)\}.$$

As a last ingredient we recall that a group representation ν on a \mathbb{C} -vector space W is said to be *defined over a subfield* $\mathbb{F} \subset \mathbb{C}$, if there exists an \mathbb{F} -vector subspace $W_{\mathbb{F}} \subset W$, stable under the group action, and such that the canonical map $W_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{C} \rightarrow W$ is an isomorphism. In this case, we say that $W_{\mathbb{F}}$ is an \mathbb{F} -*structure* for (ν, W) .

Remark 3.2. If (ν, W) is irreducible, then a rational structure is unique up to homothety, if it exists. Moreover, if $W_{\mathbb{F}}$ is an \mathbb{F} -structure for (ν, W) , with (ν, W) irreducible, and if V is a complex vector space with a trivial group action then any \mathbb{F} -structure for $(\nu \otimes 1, W \otimes V)$ is of the form $W_{\mathbb{F}} \otimes V_{\mathbb{F}}$ for a unique \mathbb{F} -structure $V_{\mathbb{F}}$ of V (as a complex vector space).

It is easy to see that as a representation of $G(E)$, E_μ has a $\mathbb{Q}(E_\mu)$ -structure, whence, so does $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$, cf. [Clo90], p. 122.

Proposition 3.3. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_E)$. Then Π_f has a $\mathbb{Q}(\Pi_f)$ -structure, which is unique up to homotheties. If Π is cohomological with respect to E_μ , then $\mathbb{Q}(\Pi_f)$ is a number field. Similarly, $H^q(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu)$ has a $\mathbb{Q}(\Pi_f)$ -structure coming from the natural $\mathbb{Q}(E_\mu)$ -structure of $H^q(\mathcal{S}_E, \mathcal{E}_\mu)$.*

Proof. This is contained in [Clo90], Prop. 3.1, Thm. 3.13 and Prop. 3.16 (the Drinfeld-Manin principle). The reader may also have a look at [GR14] Thm. 8.1 and Thm. 8.6. For the last statement, one observes that $\mathbb{Q}(E_\mu) \subseteq \mathbb{Q}(\Pi_f)$ by Strong Multiplicity One (Cf. [GR14], proof of Cor. 8.7.). \square

4. THE WHITTAKER CRSs

4.1. Rational structures on Whittaker models. We recall the discussion of [RS08], §3.2, resp. [Mah05], §3.3. Fix a non-archimedean place w of E . Given a Whittaker function ξ on $G(E_w)$ and $\sigma \in \text{Aut}(\mathbb{C})$ we define the Whittaker function ${}^\sigma\xi$ by

$$(4.1) \quad {}^\sigma\xi(g) := \sigma(\xi(t_\sigma \cdot g)),$$

where t_σ is the (unique) element in $T(E_w) \cap \mathcal{P}(E_w)$ which conjugates ψ_U to $\sigma\psi_U$. Note that t_σ does not depend on ψ . We have, $t_{\sigma_1\sigma_2} = t_{\sigma_1}t_{\sigma_2}$ and hence ${}^{\sigma_1\sigma_2}\xi \equiv \sigma_1({}^{\sigma_2}\xi)$ for all $\sigma_1, \sigma_2 \in \text{Aut}(\mathbb{C})$. Thus, if π is any irreducible admissible generic representation of $G(E_w)$,

then we obtain a σ -linear intertwining operator $\mathcal{T}_\sigma^\psi : \mathcal{W}^\psi(\pi) \rightarrow \mathcal{W}^\psi(\sigma\pi)$. In particular, we get a $\mathbb{Q}(\pi)$ structure on $\mathcal{W}^\psi(\pi)$ by taking invariant vectors. A similar discussion applies to irreducible admissible generic representations of $G(\mathbb{A}_{E,f})$.

4.2. The map $W^\psi : \Pi \rightarrow \mathcal{W}^\psi(\Pi)$ gives rise to an isomorphism

$$(4.2) \quad H^q(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \xrightarrow{\sim} H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^\psi(\Pi) \otimes E_\mu) \cong H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu) \otimes \mathcal{W}^{\psi_f}(\Pi_f).$$

Recall the $\mathbb{Q}(\Pi_f)$ -structure on $H^q(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu)$, (respectively on $\mathcal{W}^{\psi_f}(\Pi_f)$) from Prop. 3.3 (respectively from §4.1). Thus, by Rem. (3.2) we obtain a $\mathbb{Q}(\Pi_f)$ -structure on the cohomology space $H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)$ (as a \mathbb{C} -vector space) which we denote by $\mathbb{S}_\Pi^{\psi,q}$ and call it the q -th *Whittaker comparison rational structure* (CRS) of Π . In particular, $\mathbb{S}_\Pi^{\psi,q}$ is a $\mathbb{Q}(\Pi_f)$ -vector subspace of $H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)$ and $\dim_{\mathbb{Q}(\Pi_f)} \mathbb{S}_\Pi^{\psi,q} = \dim_{\mathbb{C}} H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu)$.

4.3. Equivariance of local Rankin-Selberg integrals. ¹

For this subsection only let F be a local non-archimedean field. Let π_1 and π_2 be two generic irreducible representations of $\mathrm{GL}_n(F)$ with Whittaker models $\mathcal{W}^\psi(\pi_1)$ and $\mathcal{W}^{\psi^{-1}}(\pi_2)$ respectively. Given $W_1 \in \mathcal{W}^\psi(\pi_1)$, $W_2 \in \mathcal{W}^{\psi^{-1}}(\pi_2)$ and $\Phi \in \mathcal{S}(F^n)$ a Schwartz-Bruhat function, consider the zeta integral

$$Z^\psi(W_1, W_2, \Phi, s) = \int_{U(F) \backslash \mathrm{GL}_n(F)} W_1(g) W_2(g) \Phi(e_n g) |\det g|^s dg$$

where e_n is the row vector $(0, \dots, 0, 1)$ and the invariant measure dg on $U(F) \backslash \mathrm{GL}_n(F)$ is *rational*, i.e., it assigns rational numbers to compact open subsets. We view the above expression as a formal Laurent series $A^\psi(W_1, W_2, \Phi) \in \mathbb{C}((X))$ in $X = q^{-s}$ whose m -th coefficient $c_m^\psi(W_1, W_2, \Phi)$ is

$$\int_{U(F) \backslash \mathrm{GL}_n(F) : |\det g| = q^{-m}} W_1(g) W_2(g) \Phi(e_n g) dg.$$

The last integral reduces to a finite sum, and vanishes for $m \ll 0$, because of the support of Whittaker functions. It is therefore clear (by a simple change of variable) that

$$\sigma(c_m^\psi(W_1, W_2, \Phi)) = c_m^\psi(\sigma W_1, \sigma W_2, \sigma \Phi)$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. Thus,

$$A^\psi(\sigma W_1, \sigma W_2, \sigma \Phi) = (A^\psi(W_1, W_2, \Phi))^\sigma$$

where σ acts on $\mathbb{C}((X))$ in the obvious way. The linear span of

$$A^\psi(W_1, W_2, \Phi), \quad W_1 \in \mathcal{W}^\psi(\pi_1), \quad W_2 \in \mathcal{W}^{\psi^{-1}}(\pi_2), \quad \Phi \in \mathcal{S}(F^n)$$

is a fractional ideal $\mathfrak{I}^\psi(\pi_1, \pi_2)$ of $\mathbb{C}[X, X^{-1}]$. Thus, by the above,

$$\mathfrak{I}^\psi(\pi_1, \pi_2)^\sigma = \mathfrak{I}^\psi(\sigma \pi_1, \sigma \pi_2)$$

¹Essentially the same argument is given in the middle of the proof of Theorem 2 of [Gre03]. We have restated it separately for convenience.

for any $\sigma \in \text{Aut}(\mathbb{C})$. Hence, if we write $L(s, \pi_1 \times \pi_2) = (P_{\pi_1, \pi_2}(q^{-s}))^{-1}$ where $(P_{\pi_1, \pi_2}(X))^{-1}$ is the generator of $\mathfrak{I}^\psi(\pi_1, \pi_2)$ such that $P_{\pi_1, \pi_2} \in \mathbb{C}[X]$ and $P_{\pi_1, \pi_2}(0) = 1$ then it follows that $P_{\sigma\pi_1, \sigma\pi_2} = P_{\pi_1, \pi_2}^\sigma$.

Of course, this argument applies equally well to other L -factors defined by the Rankin-Selberg method.

5. A COHOMOLOGICAL INTERPRETATION OF $\text{Res}_{s=1} L(s, \Pi \times \Pi^\vee)$

5.1. A pairing. For any compact open subgroup K_f of $G(\mathbb{A}_{E,f})$ we use the de Rham isomorphism to define a canonical map of vector spaces

$$H_c^d(\mathcal{S}_{E, K_f}, \underline{\mathbb{C}}) \xrightarrow{\int_{\mathcal{S}_{E, K_f}}} \mathbb{C}.$$

Thus, if $p + q = d$ then we get a canonical non-degenerate pairing

$$(5.1) \quad H_c^p(\mathcal{S}_{E, K_f}, \mathcal{E}_\mu) \times H^q(\mathcal{S}_{E, K_f}, \mathcal{E}_\mu^\vee) \rightarrow \mathbb{C}$$

which is defined by taking the cap product to $H_c^d(\mathcal{S}_E, \mathcal{E}_\mu \otimes \mathcal{E}_\mu^\vee)$, mapping it to $H_c^d(\mathcal{S}_E, \underline{\mathbb{C}})$ using the canonical map $\mathcal{E}_\mu \otimes \mathcal{E}_\mu^\vee \rightarrow \underline{\mathbb{C}}$ and finally applying $\int_{\mathcal{S}_{E, K_f}}$. Note however that, as defined, the maps $\int_{\mathcal{S}_{E, K_f}}$ do not fit together compatibly to a map $H_c^d(\mathcal{S}_E, \underline{\mathbb{C}}) \rightarrow \mathbb{C}$, since we have to take into account the degrees of the covering maps $\mathcal{S}_{E, K_f} \rightarrow \mathcal{S}_{E, K'_f}$, $K_f \subset K'_f$. To rectify the situation, we fix once and for all a \mathbb{Q} -valued Haar measure γ on $G(\mathbb{A}_{E,f})$ (which is unique up to multiplication by \mathbb{Q}^*). The normalized integrals $\int'_{\mathcal{S}_{E, K_f}} : H_c^d(\mathcal{S}_{E, K_f}, \underline{\mathbb{C}}) \rightarrow \mathbb{C}$ given by $\int'_{\mathcal{S}_{E, K_f}} = \text{vol}_\gamma(K_f) \int_{\mathcal{S}_{E, K_f}}$ are compatible with the pull-back with respect to the covering maps $\mathcal{S}_{E, K_f} \rightarrow \mathcal{S}_{E, K'_f}$, $K_f \subset K'_f$. Thus, we get a map

$$H_c^d(\mathcal{S}_E, \underline{\mathbb{C}}) \xrightarrow{\int'_{\mathcal{S}_E}} \mathbb{C}.$$

We denote the resulting pairing

$$\mathbb{P}^p : H_c^p(\mathcal{S}_E, \mathcal{E}_\mu) \times H^q(\mathcal{S}_E, \mathcal{E}_\mu^\vee) \rightarrow \mathbb{C}$$

It depends implicitly on the choice of γ , but this ambiguity is only up to an element of \mathbb{Q}^* .² At any rate, the maps $\int_{\mathcal{S}_{E, K_f}}$ (and consequently, $\int'_{\mathcal{S}_{E, K_f}}$ and $\int'_{\mathcal{S}_E}$) are $\text{Aut}(\mathbb{C})$ -equivariant with respect to the standard rational structure of $H_c^d(\mathcal{S}_{E, K_f}, \underline{\mathbb{C}})$ and $H_c^d(\mathcal{S}_E, \underline{\mathbb{C}})$. The same is therefore true for the pairing (5.1) and \mathbb{P}^p (with respect to the rational structure of \mathcal{E}_μ).

As noted in [Clo90, p. 124], the pairing \mathbb{P}^p restricts to a non-degenerate pairing

$$H_{\text{cusp}}^p(\mathcal{S}_E, \mathcal{E}_\mu) \times H_{\text{cusp}}^q(\mathcal{S}_E, \mathcal{E}_\mu^\vee) \rightarrow \mathbb{C}$$

and therefore to a non-degenerate pairing

$$H_{\Pi_f}^p(\mathcal{S}_E, \mathcal{E}_\mu) \times H_{\Pi_f^\vee}^q(\mathcal{S}_E, \mathcal{E}_\mu^\vee) \rightarrow \mathbb{C}$$

²This point is implicit in [Clo90, p. 124].

where $H_{\Pi_f}^p(\mathcal{S}_E, \mathcal{E}_\mu)$ is the Π_f -isotypic part of $H_{\text{cusp}}^p(\mathcal{S}_E, \mathcal{E}_\mu)$ and similarly for $H_{\Pi_f}^q(\mathcal{S}_E, \mathcal{E}_\mu^\vee)$. Composing with Δ^p and Δ^q we finally get a non-degenerate pairing

$$\mathbb{K}^{\text{Pet}, p} : H^p(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \Pi^\vee \otimes E_\mu^\vee) \rightarrow \mathbb{C}.$$

This pairing coincides with the volume of \mathcal{S}_E with respect to the complex-valued measure $\mathbb{K}_{(\Pi_\infty, \Pi_\infty^\vee, (\cdot, \cdot)_{\text{Pet}})}^{\text{coh}, p} \otimes \gamma$ of $G(\mathbb{A}_E)^1/K_\infty \cong G_\infty/A_G K_\infty \times G(\mathbb{A}_{E,f})$. Here we identify $(\wedge^d \mathfrak{p})^*$ with the space of invariant measures on $G_\infty/A_G K_\infty$.

5.2. Measures over E . At this point it will be convenient to introduce some Haar measures on various groups. If w is non-archimedean we take the Haar measure on E_w which gives volume one to the integers \mathcal{O}_w . On \mathbb{C} we take twice the Lebesgue measure. Having fixed measures on E_w for all w we can define (unnormalized) Tamagawa measures on local groups by providing a gauge form (up to a sign). On the groups G , \mathcal{P} and U we take the gauge form $\wedge dx_{i,j}/(\det x)^k$ where (i, j) range over the coordinates of the non-constant entries in the group and k is n , $n-1$ and 0 respectively. Note that if w is non-archimedean then the volume of $G(\mathcal{O}_w)$ is $\Delta_{G,w}^{-1}$ where $\Delta_{G,w} = \prod_{j=1}^n L(j, \mathbf{1}_{E_w^*})$. On $G(\mathbb{A}_E)$ and $G(\mathbb{A}_{E,f})$ we will take the measure

$$\prod_w \Delta_{G,w} dg_w$$

where w ranges over all places (resp., all finite places). On A_G we take the Haar measure whose push-forward (to \mathbb{R}_+) under $|\det|_{\mathbb{A}_E^*}$ is dx/x where dx is the Lebesgue measure. The isomorphism $G(\mathbb{A}_E) \cong A_G \times G(\mathbb{A}_E)^1$ gives a measure on $G(\mathbb{A}_E)^1$. Then

$$\text{vol}(G(E) \backslash G(\mathbb{A}_E)^1) = |D_E|^{n^2/2} \text{Res}_{s=1} \prod_{j=1}^n \zeta_E^*(s+j-1)$$

where D_E is the discriminant of E and $\zeta_E^*(s)$ is the completed Dedekind zeta function of E .

Let $\xi \in (\wedge^d \tilde{\mathfrak{p}})^*$ correspond to the invariant measure on $A_G \backslash G_\infty/K_\infty$ obtained by the push-forward of the Haar measure on $A_G \backslash G_\infty$ chosen above. Let $\Lambda_0 \in \wedge^d \tilde{\mathfrak{p}}$ be the element such that $\xi(\Lambda_0) = 1$.

5.3. The Whittaker realization of the Petersson inner product. Given a finite set of places S of E and an irreducible generic essentially unitarizable representation π_S of $G(E_S)$ with Whittaker model $\mathcal{W}^{\psi_S}(\pi_S)$ define

$$[W, W^\vee]_S := \frac{\Delta_{G,S}}{L(1, \pi_S \times \pi_S^\vee)} \cdot \int_{U(E_S) \backslash \mathcal{P}(E_S)} W(g) W^\vee(g) dg, \quad W \in \mathcal{W}^\psi(\pi_S), \quad W^\vee \in \mathcal{W}^{\psi^{-1}}(\pi_S^\vee)$$

where $\Delta_{G,S} = \prod_{w \in S} \Delta_{G,w}$. It is well known that this integral converges and defines a $G(E_S)$ -invariant pairing on $\mathcal{W}^\psi(\pi_S) \times \mathcal{W}^{\psi^{-1}}(\pi_S^\vee)$. If S consists of the archimedean places of E , we simply write $[W, W^\vee]_\infty$.

We note that if S consists of non-archimedean places only, then $[W, W^\vee]_S$ is $\text{Aut}(\mathbb{C})$ -equivariant, i.e.,

$$[\sigma W, \sigma W^\vee]_S = \sigma([W, W^\vee]_S).$$

Indeed, by uniqueness, it suffices to check this relation when the restriction of W to $\mathcal{P}(E_S)$ is compactly supported modulo $U(E_S)$, in which case the integral reduces to a finite sum and the assertion follows from §4.3 and the fact that the measures chosen on $U(E_S)$ and $\mathcal{P}(E_S)$ assign rational values to compact open subgroups.

With our choice of measures, given cuspidal automorphic forms φ, φ^\vee in the space of Π and Π^\vee , respectively, we abbreviate $W_\varphi^\psi = W^\psi(\varphi)$, $W_{\varphi^\vee}^{\psi^{-1}} = W^{\psi^{-1}}(\varphi^\vee)$ and obtain

$$(5.2) \quad (\varphi, \varphi^\vee)_{\text{Pet}} = |D_E|^{n(n+1)/4} \text{Res}_{s=1} L(s, \Pi \otimes \Pi^\vee) [W_\varphi^\psi, W_{\varphi^\vee}^{\psi^{-1}}]_S$$

(see [LM15, p. 477] which is of course based on [JS81]) provided that S is a finite set of places of E containing all the archimedean ones as well as all the non-archimedean places for which either φ or φ^\vee is not $G(\mathcal{O}_w)$ -invariant or the conductor of ψ_w is different from \mathcal{O}_w . (Note that $[W_\varphi^\psi, W_{\varphi^\vee}^{\psi^{-1}}]_S$ is unchanged by enlarging S because of the extra factor $\Delta_{G,S}$ in the numerator.)

We will also write $[W, W^\vee]_f = [W, W^\vee]_S$ for any $W \in \mathcal{W}^{\psi_f}(\Pi_f)$ and $W^\vee \in \mathcal{W}^{\psi_f^{-1}}(\Pi_f^\vee)$ where S is any sufficiently large set of non-archimedean places of E (depending on W and W^\vee).

5.4. A relation between the Whittaker CRSs and $\text{Res}_{s=1} L(s, \Pi \times \Pi^\vee)$.

Theorem 5.3. *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_E) = \text{GL}_n(\mathbb{A}_E)$, which is cohomological with respect to an irreducible, finite-dimensional, algebraic representation E_μ . For all degrees p , the number $\left(|D_E|^{n(n+1)/4} \text{Res}_{s=1} L(s, \Pi \times \Pi^\vee)\right)^{-1}$ spans the one-dimensional $\mathbb{Q}(\Pi_f)$ -vector space*

$$\Lambda \circ \mathbb{K}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee), [\cdot, \cdot]_\infty)}^{\text{coh}, p} (\mathbb{S}_\Pi^{\psi, p}, \mathbb{S}_{\Pi^\vee}^{\psi^{-1}, d-p}) \subset \mathbb{C}$$

where $\Lambda : (\wedge^d \tilde{\mathfrak{p}})^* \rightarrow \mathbb{C}$ is the evaluation at the element $\Lambda_0 \in \wedge^d \tilde{\mathfrak{p}}$ defined in §5.2.

Proof. We have two pairings $\mathbb{K}_{\Pi_\infty}^{\text{local}, p}$ and $\mathbb{K}_{\Pi}^{\text{global}, p}$ on

$$H^p(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty) \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty}(\Pi_\infty^\vee) \otimes E_\mu^\vee)$$

(with $p + q = d$): Firstly, the local pairing $\mathbb{K}_{\Pi_\infty}^{\text{local}, p}$ is defined to be

$$\mathbb{K}_{\Pi_\infty}^{\text{local}, p} := \Lambda \circ \mathbb{K}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee), [\cdot, \cdot]_\infty)}^{\text{coh}, p}.$$

Secondly, in order to define $\mathbb{K}_{\Pi}^{\text{global}, p}$ we use the isomorphism (4.2): Namely, the pairing $\mathbb{K}_{\Pi}^{\text{global}, p}$ is the one which is compatible under (4.2) with the pairing $\mathbb{K}^{\text{Pet}, p}$ (defined in §5.1) on $H^p(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \times H^q(\mathfrak{m}_G, K_\infty, \Pi^\vee \otimes E_\mu^\vee)$ and the pairing $[\cdot, \cdot]_f$ on $\mathcal{W}^{\psi_f}(\Pi_f) \times \mathcal{W}^{\psi_f^{-1}}(\Pi_f^\vee)$. By (5.2) and our convention of measures we have

$$\mathbb{K}_{\Pi}^{\text{global}, p} = |D_E|^{n(n+1)/4} \text{Res}_{s=1} L(s, \Pi \times \Pi^\vee) \cdot \mathbb{K}_{\Pi_\infty}^{\text{local}, p}.$$

On the other hand, $\mathbb{K}_{\Pi}^{\text{global}, p}(\mathbb{S}_\Pi^{\psi, i}, \mathbb{S}_{\Pi^\vee}^{\psi^{-1}, d-p}) = \mathbb{Q}(\Pi_f)$. This follows from the definition of $\mathbb{S}_\Pi^{\psi, p}$ and the fact that (1) $[\cdot, \cdot]_f$ is $\mathbb{Q}(\Pi_f)$ -rational with respect to the $\mathbb{Q}(\Pi_f)$ -structures on $\mathcal{W}^{\psi_f}(\Pi_f)$

and $\mathcal{W}^{\psi_f^{-1}}(\Pi_f^\vee)$ and (2) $\mathbb{K}^{\text{Pet}, p}$ is $\mathbb{Q}(\Pi_f)$ -rational with respect to the $\mathbb{Q}(\Pi_f)$ -structures on $H^p(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu)$ and $H^q(\mathfrak{m}_G, K_\infty, \Pi^\vee \otimes E_\mu^\vee)$ (see §5.1). The theorem follows. \square

6. A COHOMOLOGICAL INTERPRETATION OF $\text{Res}_{s=1} L(s, \Pi, \text{As}^{(-1)^{n-1}})$

6.1. Locally symmetric spaces over F . We write $G'_\infty = R_{F/\mathbb{Q}}G(\mathbb{R})$, where $R_{F/\mathbb{Q}}$ denotes restriction of scalars from F to \mathbb{Q} , and denote by K'_∞ the connected component of the identity of the intersection $K_\infty \cap G'_\infty$. It is isomorphic to $\text{SO}(n)^{[F:\mathbb{Q}]}$. We write A'_G for the group of positive reals embedded diagonally in the center of G'_∞ . (It will be convenient to distinguish between the isomorphic groups A_G and A'_G .) As before, we have $G(\mathbb{A}_F) \cong G(\mathbb{A}_F)^1 \times A'_G$ where $G(\mathbb{A}_F)^1 := \{g \in G(\mathbb{A}_F) : |\det g|_{\mathbb{A}_F^*} = 1\}$. We write $\mathfrak{g}'_\infty = \text{Lie}(G'_\infty)$, $\mathfrak{k}'_\infty = \text{Lie}(K'_\infty)$, $\mathfrak{a}'_G = \text{Lie}(A'_G)$, $\mathfrak{p}' = \mathfrak{g}'_\infty / \mathfrak{k}'_\infty$ and $\tilde{\mathfrak{p}}' = \mathfrak{p}' / \mathfrak{a}'_G$.

Let

$$\mathcal{S}_F := G(F) \backslash G(\mathbb{A}_F)^1 / K'_\infty$$

be the “locally symmetric space” attached to $G(F)$. The closed (non-injective) map $\mathcal{S}_F \rightarrow \mathcal{S}_E$ gives rise to a map $H_c^q(\mathcal{S}_E, \mathcal{E}_\mu) \rightarrow H_c^q(\mathcal{S}_F, \mathcal{E}_\mu|_{\mathcal{S}_F})$ of $G(\mathbb{A}_{F,f})$ -modules.

Finally, let ϵ' be character on $G(\mathbb{A}_F)$ given by $\epsilon \circ \det$ if n is even and 1 if n is odd, where ϵ is the quadratic Hecke character associated to the extension E/F via class field theory.

As before, let Π be a cuspidal automorphic representation of $G(\mathbb{A}_E)$ which is cohomological and conjugate self-dual. Then Π is $(G(\mathbb{A}_F), \epsilon')$ -distinguished in the sense that

$$\int_{G(F) \backslash G(\mathbb{A}_F)^1} \varphi(h) \epsilon'(h) dh$$

is non-zero for some φ in Π . (Equivalently, (by [FZ95]) $L(s, \Pi, \text{As}^{(-1)^{n-1}})$ has a pole at $s = 1$.) Indeed, otherwise $L(s, \Pi, \text{As}^{(-1)^n})$ would have a pole, and hence in particular Π_∞ would be (G'_∞, χ) -distinguished where $\chi = \epsilon \circ \det$ if n is odd and $\chi = 1$ if n is even. However, it is easy to see that this is incompatible with the description of tempered distinguished representations [Pan01]. See [HL04] and [Mok12] for the relation between distinction and base change from a unitary group.

6.2. A archimedean period on cohomology. Suppose that ρ is a tempered irreducible $(\mathfrak{g}_\infty, K_\infty)$ -module which is (G'_∞, ϵ') -distinguished, i.e., there exists a non-zero (G'_∞, ϵ') -equivariant functional ℓ on ρ . (Such a functional is unique up to a constant. This follows from [AG09] and the automatic continuity in this context [BD92, vdBD88].) Observe that

$$t = \frac{n(n+1)}{2} [F : \mathbb{Q}] - 1 = \dim_{\mathbb{R}} \mathcal{S}_F = \dim_{\mathbb{R}} \tilde{\mathfrak{p}}',$$

where t is the highest degree for which $H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$ can be non-zero for a generic irreducible (essentially unitary) representation ρ . Moreover, $H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$ is one-dimensional.

Let V_λ be a highest weight representation of $\text{GL}_n(\mathbb{R})$ with parameter $\lambda = (\lambda_1, \dots, \lambda_n)$ and let $\lambda^\vee = (-\lambda_n, \dots, -\lambda_1)$. Let $(\cdot, \cdot)_\lambda$ be the standard pairing on $V_\lambda \times V_{\lambda^\vee}$. Since by assumption E_μ is conjugate self-dual, we can define a G'_∞ -invariant form $\ell_\mu : E_\mu \rightarrow \mathbb{C}$ by taking the tensor product of the pairings above over all archimedean places of E .

We define a functional

$$\mathbb{L}_{(\rho, \ell)}^{\text{coh}, t} : H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu) \rightarrow (\wedge^t \tilde{\mathfrak{p}}')^*$$

as follows. Suppose that $\tilde{\omega} \in \text{Hom}_{K_\infty}(\wedge^t \tilde{\mathfrak{p}}, \rho \otimes E_\mu)$ represents $\omega \in H^t(\mathfrak{m}_G, K_\infty, \rho \otimes E_\mu)$. We compose $\tilde{\omega}$ with the embedding $\wedge^t \tilde{\mathfrak{p}}' \hookrightarrow \wedge^t \tilde{\mathfrak{p}}$ and with $\ell \otimes \ell_\mu$ to get an element of $\mathbb{L}_{(\rho, \ell)}^{\text{coh}, t}(\omega) \in (\wedge^t \tilde{\mathfrak{p}}')^*$. We will make the following assumption:

Hypothesis 6.1. $\mathbb{L}_{(\rho, \ell)}^{\text{coh}, t}$ is non-zero.

Hopefully, this will be proved in the near future using the method of Binyong Sun (cf. [Sun13, Sun11]).

Next, we fix a \mathbb{Q} -valued Haar measure γ' on $G(\mathbb{A}_{F,f})$: As in §5.1 we use γ' to define the normalized integrals

$$H_c^t(\mathcal{S}_F, \mathbb{C}) \xrightarrow{\int_{\mathcal{S}_F, \epsilon'}'} \mathbb{C},$$

except that now we take the cup product with the class $[\epsilon'] \in H^0(\mathcal{S}_F, \mathbb{C})$ represented by ϵ' before integrating. By composing $\int_{\mathcal{S}_F, \epsilon'}'$ with the map $H_c^t(\mathcal{S}_F, \mathcal{E}_\mu|_{\mathcal{S}_F}) \rightarrow H_c^t(\mathcal{S}_F, \mathbb{C})$ induced from ℓ_μ and the map $H_c^t(\mathcal{S}_E, \mathcal{E}_\mu) \rightarrow H_c^t(\mathcal{S}_F, \mathcal{E}_\mu|_{\mathcal{S}_F})$, we get a period map $H_c^t(\mathcal{S}_E, \mathcal{E}_\mu) \rightarrow \mathbb{C}$. As before, this map is $\text{Aut}(\mathbb{C})$ -equivariant. Composing with Δ^t we finally obtain a linear form

$$\mathbb{L}^{\text{per}, t} : H^t(\mathfrak{m}_G, K_\infty, \Pi \otimes E_\mu) \rightarrow \mathbb{C}.$$

It coincides with the volume of \mathcal{S}_F with respect to the complex-valued measure $\mathbb{L}_{(\Pi_\infty, \ell_{\text{aut}})}^{\text{coh}, t} \otimes \gamma'$ of $G(\mathbb{A}_F)^1/K'_\infty \cong G'_\infty/A'_G K'_\infty \times G(\mathbb{A}_{F,f})$ where

$$\ell_{\text{aut}}(\varphi) = \int_{G(F) \backslash G(\mathbb{A}_F)^1} \varphi(h) \epsilon'(h) dh$$

and we identify an element of $(\wedge^t \tilde{\mathfrak{p}}')^*$ with an invariant measures on $G'_\infty/A'_G K'_\infty$. In particular, $\mathbb{L}^{\text{per}, t}$ is non-zero if we assume Hypothesis 6.1.

6.3. Measures over F . We now fix some measures. If v is a non-archimedean place of F , we take the Haar measure on F_v which gives volume one to the integers \mathcal{O}_v . On \mathbb{R} we take the Lebesgue measure. This gives rise to Tamagawa measures on the local groups $G(F_v)$, $\mathcal{P}(F_v)$ and $U(F_v)$ by taking the standard gauge form as in §5.2. Thus, if v is non-archimedean then the volume of $G(\mathcal{O}_v)$ is $\Delta_{G,v}^{-1}$ where $\Delta_{G',v} = \prod_{j=1}^n L(j, \mathbf{1}_{F_v^*})$. On $G(\mathbb{A}_F)$ and $G(\mathbb{A}_{F,f})$ we will take the measure

$$\prod_v \Delta_{G',v} dg_v$$

where v ranges over all places (resp., all finite places). The measure on A'_G will be determined by the isomorphism $|\det|_{\mathbb{A}_F^*} : A'_G \rightarrow \mathbb{R}_+$ and the measure dx/x on \mathbb{R}_+ where dx is the Lebesgue measure. The isomorphism $G(\mathbb{A}_F) \cong A'_G \times G(\mathbb{A}_F)^1$ gives rise to a measure on $G(\mathbb{A}_F)^1$. Then

$$\text{vol}(G(F) \backslash G(\mathbb{A}_F)^1) = |D_F|^{n^2/2} \text{Res}_{s=1} \prod_{j=1}^n \zeta_F^*(s+j-1)$$

where D_F is the discriminant of F and $\zeta_F^*(s)$ is the completed Dedekind zeta function of F .

Let $\xi' \in (\wedge^t \tilde{\mathfrak{p}}')^*$ correspond to the invariant measure on $G'_\infty/A'_G K'_\infty$ obtained by the push-forward of the Haar measure on G'_∞/A'_G chosen above. Let $\Lambda'_0 \in \wedge^t \tilde{\mathfrak{p}}'$ be the element such that $\xi'(\Lambda'_0) = 1$.

6.4. The Whittaker realization of ℓ_{aut} . Recall that ψ was a fixed character of $E \backslash \mathbb{A}_E$. We assume from now on that the restriction of ψ to $F \backslash \mathbb{A}_F$ is trivial.

Given a finite set of places S of F and an irreducible generic unitarizable $(G(F_S), \epsilon')$ -distinguished representation π_S of $G(E_S)$ with Whittaker model $\mathcal{W}^{\psi_S}(\pi_S)$ define

$$\ell_S(W) := \frac{\Delta_{G',S}}{L(1, \pi_S, \text{As}^{(-1)^{n-1}})} \cdot \int_{U(F_S) \backslash \mathcal{P}(F_S)} W(h) \epsilon'(h) dh, \quad W \in \mathcal{W}^\psi(\pi_S)$$

where $\Delta_{G',S} = \prod_{v \in S} \Delta_{G',v}$. The integral converges and defines a $(G(F_S), \epsilon')$ -equivariant form on $\mathcal{W}^{\psi_S}(\pi_S)$ ([Off11, Kem12]).

If v is a non-archimedean place of F , then by the same argument as in §4.3 we have

$$L(s, {}^\sigma \Pi_v, \text{As}^{(-1)^{n-1}}) = L(s, \Pi_v, \text{As}^{(-1)^{n-1}})^\sigma$$

(where on the right-hand side, σ acts on $\mathbb{C}(q^{-s})$ in the obvious way). In particular,

$$(6.2) \quad L(1, {}^\sigma \Pi_v, \text{As}^{(-1)^{n-1}}) = \sigma(L(1, \Pi_v, \text{As}^{(-1)^{n-1}})).$$

Thus, if S consists only of non-archimedean places, then $\ell_S(W)$ is $\text{Aut}(\mathbb{C})$ -equivariant, i.e.,

$$\ell_S({}^\sigma W) = \sigma(\ell_S(W)).$$

Indeed, by uniqueness ([Fli91]), it suffices to check this relation when the restriction of W to $\mathcal{P}(E_S)$ is compactly supported modulo $U(E_S)$, in which case the integral reduces to a finite sum and the assertion follows from (6.2) and the rationality of the measure on $U(F_S) \backslash \mathcal{P}(F_S)$.

With our choice of measures, for any cuspidal automorphic form φ in the space of Π and $W_\varphi^\psi = W^\psi(\varphi)$, we have

$$(6.3) \quad \ell_{\text{aut}}(\varphi) = |D_F|^{n(n+1)/4} \text{Res}_{s=1} L(s, \Pi, \text{As}^{(-1)^{n-1}}) \cdot \ell_S(W_\varphi^\psi)$$

provided that S is a sufficiently large finite set of places of F . (Cf. Gelbart–Jacquet–Rogawski, [GJR01], pp. 184–185 or Zhang [Zha14], Sect. 3.2.) Note that $\ell_S(W_\varphi^\psi)$ is unchanged by enlarging S because of the extra factor $\Delta_{G',S}$ in the numerator.

As before we write $\ell_f(W) = \ell_S(W)$ for $W \in \mathcal{W}^{\psi_f}(\Pi_f)$ where S is any sufficiently large set of places.

6.5. A relation between the top Whittaker CRS and $\text{Res}_{s=1} L(1, \Pi, \text{As}^{(-1)^{n-1}})$. We may now prove our first main theorem on the Asai L -function.

Theorem 6.4. *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_E) = \text{GL}_n(\mathbb{A}_E)$, which is cohomological with respect to an irreducible, finite-dimensional, algebraic representation E_μ . Assume Hypothesis 6.1. Then, $\left(|D_F|^{n(n+1)/4} \text{Res}_{s=1} L(s, \Pi, \text{As}^{(-1)^{n-1}})\right)^{-1}$ spans the one-dimensional $\mathbb{Q}(\Pi_f)$ -vector space*

$$\Lambda' \circ \mathbb{L}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \ell_\infty)}^{\text{coh}, t}(\mathbb{S}_\Pi^{\psi, t}) \subset \mathbb{C}$$

where $\Lambda' : (\wedge^t \tilde{\mathfrak{p}}')^* \rightarrow \mathbb{C}$ is the evaluation at the element $\Lambda'_0 \in \wedge^t \tilde{\mathfrak{p}}'$ defined above.

Proof. Arguing as in the proof of Thm. 5.3, the result follows using relation (6.3), which compares the global period-map $\mathbb{L}^{\text{per},t}$ with the local period-map $\Lambda' \circ \mathbb{L}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \ell_\infty)}^{\text{coh},t}$, and the fact that $\ell_f(\sigma W) = \sigma(\ell_f(W))$ for all $\sigma \in \text{Aut}(\mathbb{C})$. \square

7. A RELATION BETWEEN THE BOTTOM WHITTAKER CRS AND $L(1, \Pi, \text{As}^{(-1)^n})$

7.1. We will now put the contents of §5 and §6 together, in order to obtain a rationality result for $L(1, \Pi, \text{As}^{(-1)^n})$. To that end, recall the pairing $\mathbb{K}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee), [\cdot, \cdot]_\infty)}^{\text{coh},i}$ from §2.4. We use it to identify $\Lambda' \circ \mathbb{L}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \ell_\infty)}^{\text{coh},t}$ as an element of

$$H^b(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee) \otimes E_\mu^\vee) \otimes \wedge^{d\tilde{\mathfrak{p}}},$$

where $b = \frac{n(n-1)}{2}[F : \mathbb{Q}]$. (Recall that $b+t = d$.) We write this element as $\mathbb{M}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \ell_\infty)}^{\text{coh},b} \otimes \Lambda_0$, where

$$\mathbb{M}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \ell_\infty)}^{\text{coh},b} \in H^b(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee) \otimes E_\mu^\vee).$$

We may now prove our second main theorem on the Asai L -function.

Theorem 7.1. *Let Π be a conjugate self-dual, cuspidal automorphic representation of $G(\mathbb{A}_E) = \text{GL}_n(\mathbb{A}_E)$, which is cohomological with respect to an irreducible, finite-dimensional, algebraic representation E_μ . Assume Hypothesis 6.1. Then,*

$$\left(|D_E/D_F|^{n(n+1)/4} L(1, \Pi, \text{As}^{(-1)^n}) \right)^{-1} \cdot \mathbb{M}_{(\mathcal{W}^{\psi_\infty}(\Pi_\infty), \ell_\infty)}^{\text{coh},b}$$

spans the one-dimensional $\mathbb{Q}(\Pi_f)$ -vector subspace $\mathbb{S}_{\Pi^\vee}^{\psi^{-1},b}$ of $H^b(\mathfrak{m}_G, K_\infty, \mathcal{W}^{\psi_\infty^{-1}}(\Pi_\infty^\vee) \otimes E_\mu^\vee)$.

Proof. The theorem follows readily from Thm. 5.3 and Thm. 6.4 and the relation $L(s, \Pi \times \Pi^\tau) = L(s, \Pi, \text{As}^{(-1)^{n-1}}) \cdot L(s, \Pi, \text{As}^{(-1)^n})$. \square

Remark 7.2. Theorem 7.1 generalizes a result of the first two named authors, see [GH13] Thm. 6.22.

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